

Rank and randomness

Rupert Hölzl



Bundeswehr University Munich

Joint work with Christopher P. Porter



Drake University, Des Moines

1

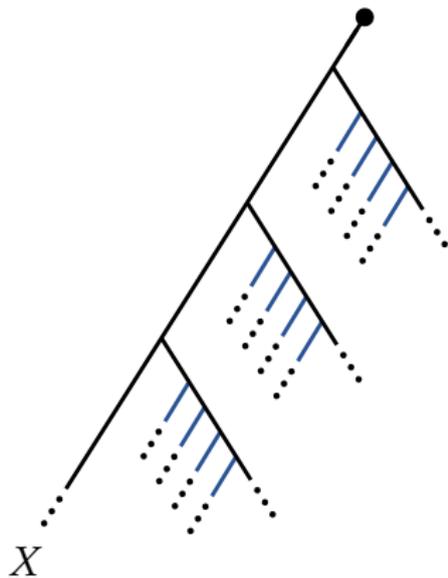
Motivation

Cantor-Bendixson rank of a sequence

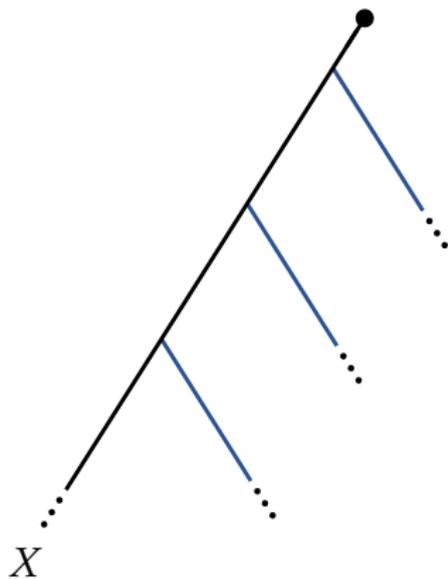
“The Cantor-Bendixson rank $\text{rk}_{\mathcal{P}}(X)$ of a sequence X inside a Π_1^0 class $\mathcal{P} \subseteq 2^\omega$ measures how tightly integrated X is into \mathcal{P} .”

- 1 Formally:** Iterate the operation of removing all isolated sequences from \mathcal{P} . Then (if it exists) $\text{rk}_{\mathcal{P}}(X)$ is the number of the last iteration after which X is still present.

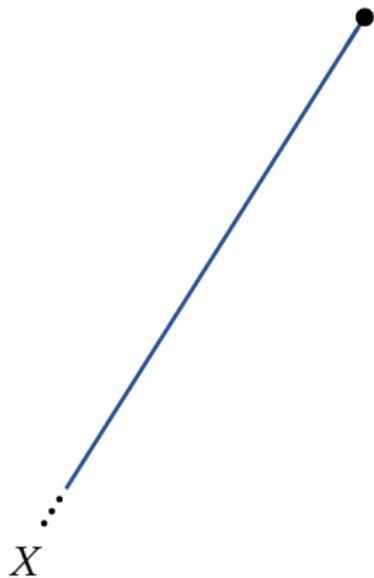
Examples



Examples

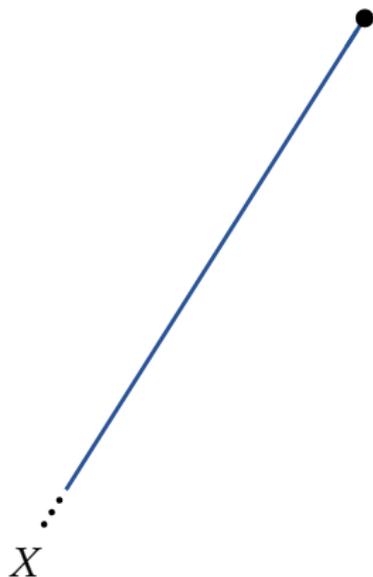


Examples



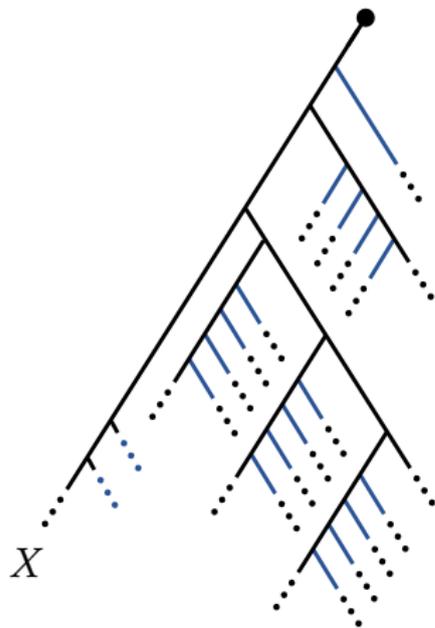
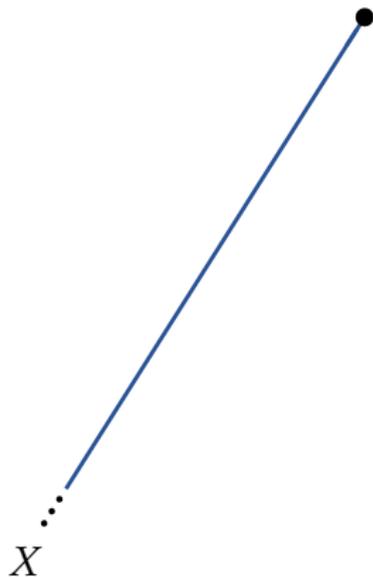
X is now isolated

Examples



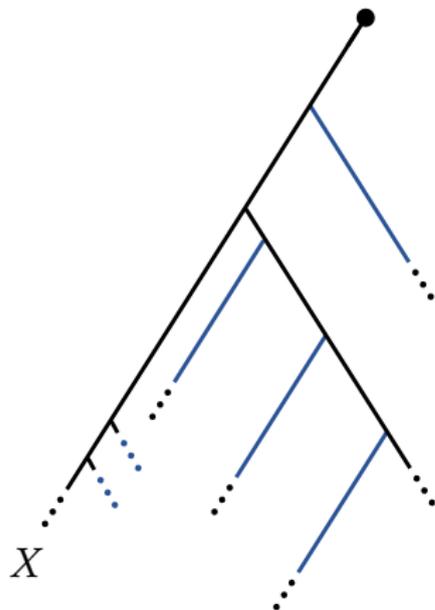
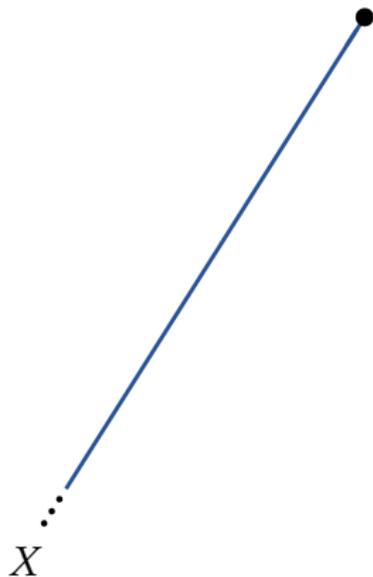
X is now isolated
 $\Rightarrow \text{rk}_{\mathcal{D}}(X) = 2.$

Examples



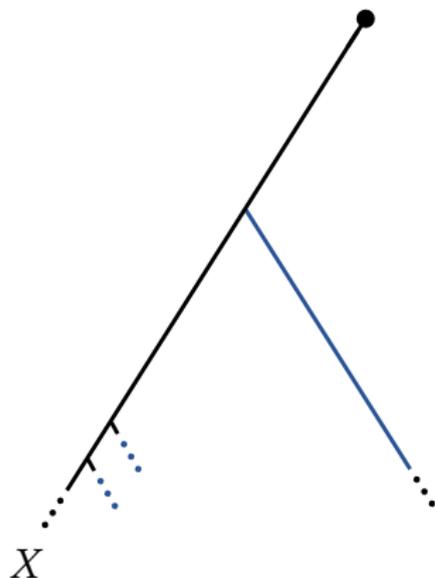
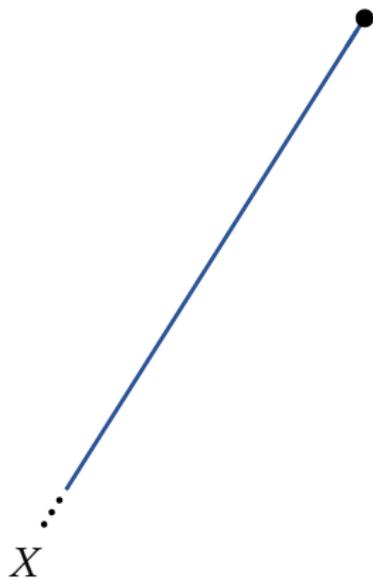
X is now isolated
 $\Rightarrow \text{rk}_{\mathcal{P}}(X) = 2.$

Examples



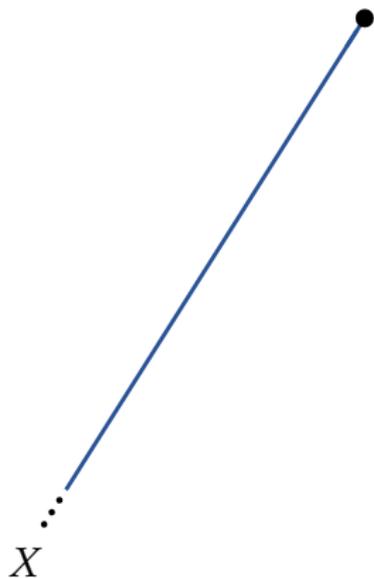
X is now isolated
 $\Rightarrow \text{rk}_{\mathcal{P}}(X) = 2.$

Examples

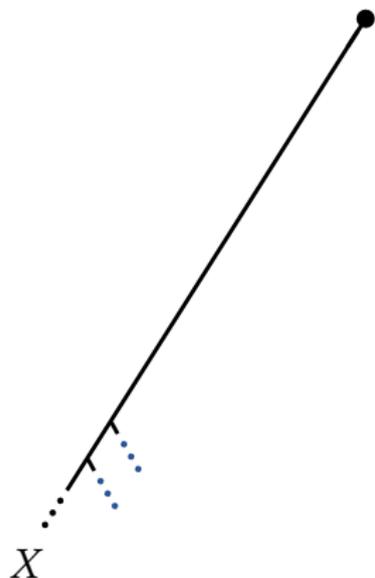


X is now isolated
 $\Rightarrow \text{rk}_{\mathcal{P}}(X) = 2.$

Examples

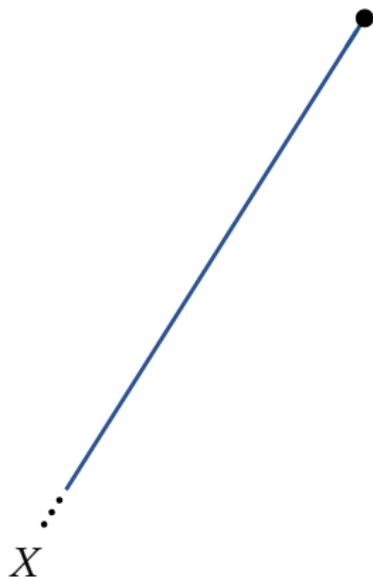


X is now isolated
 $\Rightarrow \text{rk}_{\mathcal{P}}(X) = 2.$

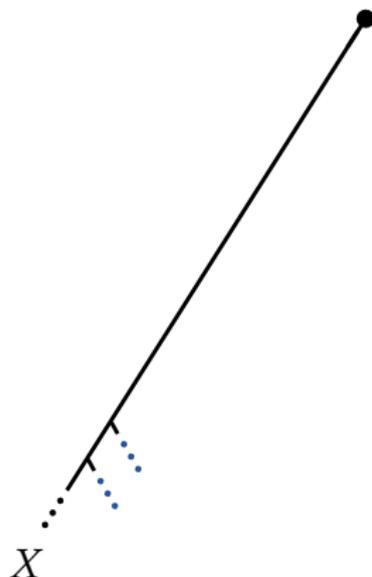


X is still not isolated

Examples



X is now isolated
 $\Rightarrow \text{rk}_{\mathcal{P}}(X) = 2.$



X is still not isolated
 $\Rightarrow \text{rk}_{\mathcal{P}}(X) = \omega.$

- 1 Definition.** If a Π_1^0 class \mathcal{P} exists where $\text{rk}_{\mathcal{P}}(X)$ is defined, then

$$\text{rk}(X) = \min\{\alpha: \mathcal{Q} \text{ is a } \Pi_1^0 \text{ class with } \text{rk}_{\mathcal{Q}}(X) = \alpha\}.$$

Otherwise $\text{rk}(X)$ is undefined.

- 2** By a result of Kreisel, if it exists, $\text{rk}(X)$ is a computable ordinal.
- 3 Theorem (Cenzer, Smith).** For all computable ordinals $\alpha > 0$ there is some Δ_2^0 degree \mathbf{a} such that some $X \in \mathbf{a}$ has $\text{rk}(X) = \alpha$.
- 4 Theorem (Cenzer, Smith).** Every Δ_2^0 degree has a rank 1 point.
- 5 Theorem (Downey, Wu, Yang).** For every Δ_2^0 degree \mathbf{a} and every computable ordinal $\alpha > 0$, exists an $X \in \mathbf{a}$ with $\text{rk}(X) = \alpha$.

- 1 **Definition.** If a Π_1^0 class \mathcal{P} exists where $\text{rk}_{\mathcal{P}}(X)$ is defined, then

$$\text{rk}(X) = \min\{\alpha: \mathcal{Q} \text{ is a } \Pi_1^0 \text{ class with } \text{rk}_{\mathcal{Q}}(X) = \alpha\}.$$

Otherwise $\text{rk}(X)$ is undefined.

- 2 By a result of Kreisel, if it exists, $\text{rk}(X)$ is a computable ordinal.
- 3 **Theorem (Cenzer, Smith).** For all computable ordinals $\alpha > 0$ there is some Δ_2^0 degree \mathbf{a} such that some $X \in \mathbf{a}$ has $\text{rk}(X) = \alpha$.
- 4 **Theorem (Cenzer, Smith).** Every Δ_2^0 degree has a rank 1 point.
- 5 **Theorem (Downey, Wu, Yang).** For every Δ_2^0 degree \mathbf{a} and every computable ordinal $\alpha > 0$, exists an $X \in \mathbf{a}$ with $\text{rk}(X) = \alpha$.
- 6 **Question.** Is there a relation between rank and *randomness*?

- 1 Proposition.** If $\text{rk}(X)$ exists, X is inside a countable Π_1^0 class.
- 2** In particular, for no $X \in \text{MLR}$ does $\text{rk}(X)$ exist.

- 1 Proposition.** If $\text{rk}(X)$ exists, X is inside a countable Π_1^0 class.
- 2** In particular, for no $X \in \text{MLR}$ does $\text{rk}(X)$ exist.
- 3 Proposition (Porter, Bienvenu).** There is an X with $\text{rk}(X) = 1$ such that X is random with respect to a computable measure.
(Note that such a measure must have atoms, but that X cannot be one of them.)
- 4 Proposition (Porter).** In every Δ_2^0 random degree exists an X random with respect to a computable measure with $\text{rk}(X) = 2$.

- 1 **Proposition.** If $\text{rk}(X)$ exists, X is inside a countable Π_1^0 class.
- 2 In particular, for no $X \in \text{MLR}$ does $\text{rk}(X)$ exist.
- 3 **Proposition (Porter, Bienvenu).** There is an X with $\text{rk}(X) = 1$ such that X is random with respect to a computable measure.
(Note that such a measure must have atoms, but that X cannot be one of them.)
- 4 **Proposition (Porter).** In every Δ_2^0 random degree exists an X random with respect to a computable measure with $\text{rk}(X) = 2$.
- 5 **Our goal:** A stronger and more general result, analogous to that of Downey/Wu/Yang, but for randomness.

Our main result

1 Definition. A Π_1^0 class \mathcal{P} is *rank-faithful* if $\text{rk}(X) = \text{rk}_{\mathcal{P}}(X)$ for all $X \in \mathcal{P}$ such that $\text{rk}(X)$ exists.

2 Definition. The *support* of a measure μ is the set

$$\{X \in 2^\omega : \forall n \mu(X \upharpoonright n) > 0\}.$$

3 Theorem. For each Δ_2^0 random degree \mathbf{r} and each computable ordinal $\alpha > 0$, there is a countably supported computable measure μ and a sequence $R \in \mathbf{r}$ such that

- R has Cantor-Bendixson rank α ,
- R is random with respect to μ , and
- the support of μ is rank-faithful.

4 Can be seen as a “randomized version” of Downey/Wu/Yang.

5 Additionally, the last point has a strong implication...

The relevance of rank-faithfulness

- 1 Let \mathcal{P} denote the rank-faithful support of μ .
- 2 Let $(\mathcal{U}_i)_{i \in \omega}$ be a universal μ -test and $\mathcal{K}_i = 2^\omega \setminus \mathcal{U}_i$ for all $i \in \omega$.
- 3 Clearly $\mathcal{K}_i \subseteq \mathcal{P}$ for every $i \in \omega$.
- 4 Then $\text{rk}_{\mathcal{K}_i}(R) \leq \text{rk}_{\mathcal{P}}(R)$ for every $i \in \omega$ and $R \in \mathcal{K}_i$.
- 5 Since $\text{rk}(R) = \text{rk}_{\mathcal{P}}(R)$ by the rank-faithfulness of \mathcal{P} , we have $\text{rk}_{\mathcal{K}_i}(R) = \text{rk}(R)$ for every $i \in \omega$ such that $R \in \mathcal{K}_i$.
- 6 This implies for every $i \in \omega$ with $R \in \mathcal{K}_i$, that R 's rank $\text{rk}(R) = \text{rk}_{\mathcal{K}_i}(R)$ can already be observed inside \mathcal{K}_i .
- 7 Since \mathcal{K}_i contains only μ -random sequences, the entire structure of sequences that give R its (minimal) rank consists of μ -randoms.
- 8 So not only have we built a μ -random R of desired rank, but the rank of R can be witnessed entirely by sequences that are μ -random themselves (and which inductively have the same property).

2

Proving the finite case

The base case $\alpha = 1$

- 1 Let $A \in \mathbf{r}$ be random and $(A_s)_{s \in \omega}$ a Δ_2^0 -approximation of A .
- 2 For $X \in 2^\omega$, define an X -computable $(s_n^X)_{n \geq 1}$ via
 - s_1^X is the least s such that $X \upharpoonright 1 = A_s \upharpoonright 1$, and
 - for $n \geq 1$, by s_{n+1}^X is the least $s > s_n^X$ with $X \upharpoonright (n+1) = A_s \upharpoonright (n+1)$, if these values exist.
- 3 Next define $\sigma_1^X, \sigma_2^X, \dots$ via

$$\sigma_n^X = \begin{cases} 1^{s_n^X} \frown 0 & \text{if } s_n^X \downarrow, \\ \uparrow & \text{otherwise.} \end{cases}$$

The base case $\alpha = 1$

- 1 Finally define the Turing functional $\Phi_A(X)$ via

$$\Phi_A(X) = \begin{cases} \sigma_1^X \sigma_2^X \dots & \text{if } \forall i \geq 0 : \sigma_i^X \downarrow, \\ \sigma_1^X \sigma_2^X \dots \sigma_{i-1}^X 1^\omega & \text{if } i \text{ is least such that } \sigma_i^X \uparrow. \end{cases}$$

- 2 The measure μ induced by Φ_A is clearly countably supported.
- 3 It's easy to see that $\Phi_A(A) \equiv_T A$, so $\Phi_A(A) \in \mathbf{r}$.
- 4 It's also immediate that for $X \neq A$, $\Phi_A(X)$ is computable.
- 5 All $\Phi_A(X)$ with $X \neq A$ are isolated, so $\text{rk}_{\Phi_A(2^\omega)}(\Phi_A(X)) = 0$.
- 6 There are infinitely many $\Phi_A(X)$ branching off $\Phi_A(A)$, so $\text{rk}_{\Phi_A(2^\omega)}(\Phi_A(A)) = 1$.
- 7 It remains to show that $\Phi_A(2^\omega)$ is rank-faithful.

From $\alpha = 1$ to $\alpha < \omega$

1 Claim. $\text{rk}(X) = \text{rk}_{\Phi_A(2^\omega)}(X)$ for all $X \in \Phi_A(2^\omega)$.

Proof. $\Phi_A(A)$ is non-computable, so there is no Π_1^0 class \mathcal{Q} in which $\Phi_A(A)$ is isolated, so $\text{rk}(\Phi_A(A)) = 1 = \text{rk}_{\Phi_A(2^\omega)}(\Phi_A(A))$.

For $\Phi_A(X)$ with $X \neq A$, trivially $\text{rk}(X) = \text{rk}_{\Phi_A(2^\omega)}(X) = 0$.

2 Note how this argument only works for $\alpha = 1$!

3 This completes the proof for $\alpha = 1$. □

From $\alpha = 1$ to $\alpha < \omega$

1 Claim. $\text{rk}(X) = \text{rk}_{\Phi_A(2^\omega)}(X)$ for all $X \in \Phi_A(2^\omega)$.

Proof. $\Phi_A(A)$ is non-computable, so there is no Π_1^0 class \mathcal{Q} in which $\Phi_A(A)$ is isolated, so $\text{rk}(\Phi_A(A)) = 1 = \text{rk}_{\Phi_A(2^\omega)}(\Phi_A(A))$.
For $\Phi_A(X)$ with $X \neq A$, trivially $\text{rk}(X) = \text{rk}_{\Phi_A(2^\omega)}(X) = 0$.

2 Note how this argument only works for $\alpha = 1$!

3 This completes the proof for $\alpha = 1$. □

4 To obtain the case $\alpha < \omega$, interleave finitely many functionals:

$$\Psi_A(X) = \bigoplus_{i=0}^{n-1} \Phi_{A_i}(X_i) \text{ where } A = \bigoplus_{i=0}^{n-1} A_i \text{ and } X = \bigoplus_{i=0}^{n-1} X_i$$

5 Then we (essentially) need to argue

- why this produces the needed ranks $\text{rk}_{\Psi_A(2^\omega)}$,
- why $\Psi_A(2^\omega)$ is rank-faithful.

The case $\alpha < \omega$

- 1 Note that if some $X_i \neq A_i$, then $\Phi_{A_i}(X_i)$ becomes computable.
- 2 If X disagrees from A in some components, then $\Psi_A(X)$ branches off of $\Psi_A(A)$, with some components becoming computable sequences.
- 3 If another sequence Y
 - initially agrees with X for a long enough time so that $\Psi_A(Y)$, branches off of $\Psi_A(A)$ at the same location as $\Psi_A(X)$
 - but then starts disagreeing from A in 1 *more* component than X , then $\Psi_A(Y)$ has 1 more computable component than $\Psi_A(X)$.
- 4 If we keep repeating this, eventually only computable components are left, resulting in a rank 0 point.
- 5 Infinitely many such rank 0 points branch off of some $\Psi_A(Z)$ (that has 1 non-computable component), giving it rank 1.
- 6 This way, the rank structure is built up to the top $\Psi_A(A)$, e.g. the $\Psi_A(Y)$'s witness the 1 higher rank of $\Psi_A(X)$, and so on.

The case $\alpha < \omega$

- 1 It remains to show the rank-faithfulness.
- 2 We use induction over n , the number of interleaved functionals.
- 3 The components of the input and the output are Turing-equivalent component-wise, that is, $\Phi_{A_i}(A_i) \equiv_T A_i$ for each i .
- 4 **Theorem (Owings).** $\text{rk}(X \oplus Y) = \text{rk}(X)$ implies $Y \leq_T X$.
- 5 But as A was random, all A_i are pairwise relatively random.
- 6 So using induction we obtain that the output rank increases by 1 with the addition of every new component A_i . \square

3

Proving the case $\alpha = \omega$

The challenge

- 1 The naïve approach (of interleaving infinitely many functionals) fails.
- 2 **Reason:** $\text{rk}_{\Psi_A(2^\omega)}(\Psi_A(A))$ would not exist, as
 - an X has ω many components X_i in which it could differ from A ;
 - so if an arbitrary finite number of components X_i has already been fixed to differ from A_i , there is always yet another component in which additional disagreement is possible;
 - this would result in yet another branching; and
 - the support of the induced measure would contain a perfect set.

The approach

- 1 We need a process that leads to infinite interleaving only if the input is A , but which otherwise results in finite interleaving.
- 2 The finite number of interleaved sequences should grow with the number of components of the input agreeing with A .

Dynamic partitioning



1 Let $J_0 = \omega$.

Dynamic partitioning



- 1 Let $J_0 = \omega$.
- 2 Partition ω into the sets
 - $I_0 = \{n \in \omega : n \equiv 0 \pmod{2}\}$,
- 3 Let $J_{n+1} = \overline{I_0 \cup \dots \cup I_n}$ for all $n \in \omega$.

Dynamic partitioning



- 1 Let $J_0 = \omega$.
- 2 Partition ω into the sets
 - $I_0 = \{n \in \omega : n \equiv 0 \pmod{2}\}$,
 - $I_1 = \{n \in \omega : n \equiv 1 \pmod{4}\}$,
- 3 Let $J_{n+1} = \overline{I_0 \cup \dots \cup I_n}$ for all $n \in \omega$.

Dynamic partitioning



1 Let $J_0 = \omega$.

2 Partition ω into the sets

- $I_0 = \{n \in \omega : n \equiv 0 \pmod{2}\}$,
- $I_1 = \{n \in \omega : n \equiv 1 \pmod{4}\}$,
- $I_2 = \{n \in \omega : n \equiv 3 \pmod{8}\}$,

3 Let $J_{n+1} = \overline{I_0 \cup \dots \cup I_n}$ for all $n \in \omega$.

Dynamic partitioning



1 Let $J_0 = \omega$.

2 Partition ω into the sets

- $I_0 = \{n \in \omega : n \equiv 0 \pmod{2}\}$,
- $I_1 = \{n \in \omega : n \equiv 1 \pmod{4}\}$,
- $I_2 = \{n \in \omega : n \equiv 3 \pmod{8}\}$,
- $I_3 = \{n \in \omega : n \equiv 7 \pmod{16}\}$,
- \vdots

3 Let $J_{n+1} = \overline{I_0 \cup \dots \cup I_n}$ for all $n \in \omega$.

Dynamic output locations

Idea:

- 1 If currently $n + 1$ functionals are active,
 - then for $i \leq n$, the i -th functional is restricted to I_i ,
 - while the $(n + 1)$ -th functional is restricted to J_{n+1} .
- 2 If ever another functional becomes active, then from that moment on the $(n + 1)$ -th functional is restricted to I_{n+1} .

Dynamic output locations

Idea:

- 1 If currently $n + 1$ functionals are active,
 - then for $i \leq n$, the i -th functional is restricted to I_i ,
 - while the $(n + 1)$ -th functional is restricted to J_{n+1} .
- 2 If ever another functional becomes active, then from that moment on the $(n + 1)$ -th functional is restricted to I_{n+1} .

Formally:

- 1 The dynamic join functional Ξ_A on input X launches $\Phi_{A_0}(X_0)$ and writes its output bits into places in $J_0 = \omega$ of the output.
- 2 Once $\Phi_{A_0}(X_0)$ “has made progress” (= outputs a 0), it is restricted to outputting its bits into the unused places of I_0 .
- 3 Then $\Phi_{A_1}(X_1)$ is launched in parallel, outputting its bits into so far unused locations of J_1 .
- 4 Once both $\Phi_{A_0}(X_0)$ and $\Phi_{A_1}(X_1)$ have made progress, $\Phi_{A_1}(X_1)$ is restricted to I_1 , and $\Phi_{A_2}(X_2)$ is launched writing into J_2 , etc.

Dynamic output locations

- 1 Note how the construction ensures that all places of the output sequence become defined, independently of whether all parallel processes continue to make progress or some get stuck.

Thus, Ξ_A is total.

- 2 If $X = A$, then all processes will always make more progress, and all countably many functionals will become launched, and all will contribute to the output sequence.
- 3 If $X \neq A$ then the last process that is launched is determined by what's the first component of X that disagrees with A (and by how soon this disagreement is found).

Properties of the dynamic join

- 1 If during the computation of $\Xi_A(X)$ the n -th process is the last that is started, then $\Xi_A(X)$ is essentially a join of $\Phi_{A_i}(X_i)$, $i \leq n$, (with the bits initially distributed in a somewhat complicated fashion).
- 2 We still have that the input and output of Ξ are Turing-equivalent *individually per component*, as for $\alpha < \omega$. (But *how many* components have been encoded into $\Xi(X)$ depends on X .)

Proof of the case $\alpha = \omega$

- 1 Assume for $X \neq A$, X agrees with A in exactly n components.
- 2 Then $\Xi_A(X)$ agrees in exactly n components with $\Xi_A(A)$.
- 3 In particular, $\Xi_A(X)$ has rank n .
(The argument is essentially the same as for $\alpha < \omega$; the finitely many “out of place” bits at the start of each component have no bearing on the infinitary notion of rank.)
- 4 Since now $\Xi_A(A)$ has sequences branching off it whose ranks are all $< \omega$ but converge to ω , $\Xi_A(A)$ itself must have rank ω .
- 5 Finally, rank-faithfulness is essentially shown as for $\alpha < \omega$. \square

4

Proof of the general case

Proof of the general case

- 1 The proof is by effective transfinite induction; some overhead is needed to handle notations for arbitrary computable ordinals.
- 2 Otherwise the proof is quite similar:
 - The successor case is similar to the inductive step for $\alpha < \omega$.
 - The limit case is similar to the proof for the case $\alpha = \omega$. □

- 1 **Theorem, restated.** For each Δ_2^0 random degree \mathbf{r} and each computable ordinal $\alpha > 0$, there is a countably supported computable measure μ and a sequence $R \in \mathbf{r}$ such that
- R has Cantor-Bendixson rank α ,
 - R is random with respect to μ , and
 - the support of μ is rank-faithful.

- 1 **Theorem, restated.** For each Δ_2^0 random degree \mathbf{r} and each computable ordinal $\alpha > 0$, there is a countably supported computable measure μ and a sequence $R \in \mathbf{r}$ such that
- R has Cantor-Bendixson rank α ,
 - R is random with respect to μ , and
 - the support of μ is rank-faithful.

Thank you for your attention.

arxiv.org/abs/1707.00378